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MANDELBROT SET, THE MESMERIZING FRACTAL WITH INTEGER DIMENSION

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ABSTRACT

In this paper we have given a brief review of the Mandelbrot set, one of the best known icons of fractals which arises from the iteration of the complex polynomial of the form $z^2 + c$. We have discussed about the role of critical points in such a study with the help of Schwarzian derivative.

KEYWORDS: Critical Orbit, Fixed Point, Iteration of a Map, Julia Set, Mandelbrot Set, Periodic Point, Schwarzian Derivative

1. INTRODUCTION

The Mandelbrot set has a celebrated place in fractal geometry, a field first investigated by the French mathematicians Gaston Julia and Pierre Fatou as a part of complex dynamics at the beginning of the 20th century. Gaston Julia(1893-1978) wrote a paper titled "*M´emoire sur l'iteration des fonctions rationelles*" (A Note on the Iteration of Rational Functions) **[**10] where he first introduced the modern idea of a Julia set as a part of complex dynamics. In this paper Julia gave a precise description of the set of those points of the complex plane whose orbits under the iteration of a rational function stayed bounded. In 1978 Robert W. Brooks and Peter Matelski investigated some subgroups of Kleinian groups [17] and as a part of this investigation they first introduced the concept of what we now call Mandelbrot set.

Benoit Mandelbrot (1924-2010) was a Polish-born French mathematician, who spent most of his career at IBM's Thomas J. Watson Research Center in Yorktown Height, New York. He was inspired by Julia's above mentioned paper on complex dynamics and used computers to explore these works. In the year 1977, as a result of his research, he discovered one of the most famous fractals, which now bears his name: the Mandelbrot set. On 1st March 1980 Mandelbrot first visualized this set [18]. He studied the parameter space of the complex quadratic polynomials in an article that appeared in the '*Annals of New York Academy of science*'[12].

The Mathematical study of the Mandelbrot set actually began with the works of Adrien Douady and John H. Hubbard [7] who established many of its fundamental properties and named the set in honor of Mandelbrot. Interest in the subject flourished over and many other well known mathematicians began to study the Mandelbrot set. Heinz-Otto Peitgen and Peter Richter are the name of two such mathematicians who became well known for promoting the Mandelbrot set with computer oriented graphics and books [15]. A good account of developing period of the theory of complex dynamics can be found in [2],[5],[6],[22]. The authors are among the most active contributors to this field.

Mandelbrot set may well be one of the most familiar images produced by the mathematicians and other related scientists of the 20^{th} century. It challenges the familiar notion that the domain of simplicity and complexity are entirely different. Because, the mathematical formula that is involved in the construction of Mandelbrot set consists of simple operations like multiplication and addition, still it produces a shape of great organic beauty and complexity with infinite subtle variations. The developments arising from the Mandelbrot set have been as diverse as the alluring shapes it generates.

The shape of the Galaxies broke all Euclidean laws of the man-made world and deferred from the properties of natural world. If one identified an essential structure like this, Mandelbrot claimed, that the concept of Mandelbrot set, in general fractal geometry, could be applied to understand its component parts and make postulations about what it will become in future. For instance interested reader may see [19] for study about distribution of galaxies in an observed universe. In today's world of wireless communication many wireless devices use fractal based compact and potable antennas that pick up the widest range of known frequencies [1], [20]. Fractal art is a form of algorithmic art created by fractal objects produced by repeated iterations of some mathematical rules and representing the calculated results as still images, animations etc. The Mandelbrot set can be considered as a great icon for fractal art. Graphic design and image editing programs use fractal to create beautifully complex landscapes and life-like special effects. Interested readers can go through [3],[16] for finding such applications. Fractal statistical analyses of forest can measure and quantify how much carbon dioxide the world can safely process [14]. Fractal geometry may also be applied to the various fields of medicine such as cardiovascular system, neurobiology, pathology and molecular biology [4], [9].

The Mandelbrot set, like most of the other fractals, arises from a simple iterative process. The process involved here is the iteration of the non-linear relation $z_{n+!} = {z_n}^2 + c$ on the points of the complex plane. It turns out that the same relation was already studied in the early 20^{th} century by France mathematicians Gaston Julia and Pierre Fatou which lead to the discovery of the Julia sets. Like the Mandelbrot set, the Julia set also have a fractal structure and are generated by using the same iterative process employed in the generation of the Mandelbrot set but with slightly different initial conditions. Interested reader may go through [11]. There is only one Mandelbrot set and infinitely many Julia sets- each point on the complex plane acting as a parameter to the Julia set.

The Mandelbrot set, a very beautiful fractal structure enjoys a special status as a cultural icon. Also, deep mathematics underlies the Mandelbrot set. Despite years of study by brilliant mathematicians, some natural and simple-to-state questions still remains un-answered. For example, though Mandelbrot set was known to the mathematical community since 1977 due to the complex form of shape its area was able to estimated to be $1.0565918849 \pm 0.0000000028$ by Thorsten Förstemann just in 2012 [8]. Much of the re-birth of interest in complex dynamics was motivated by efforts to understand the stunning images of Mandelbrot set, which is the prime objective of this paper.

The rest of the paper is organized as follows: in section 2, we provide a review of preliminary concepts and definitions. Section 3 deals with role of critical orbit in determining the dynamics of the map $f_c(z) = z^2 + c$. In section 4, we have given a brief discussion why the Mandelbrot set is a fractal of typical nature. Finally, in section 5, we have given a concluding remark of our study.

2. SOME PRELIMINARIES

In order to carry our study, we first need to provide some definitions concerning classical deterministic chaotic dynamical systems which are discussed in this section.

Definition 2.1: The *orbit* of a number z_0 under function $f:\hat{C} \to \hat{C}$ where $\hat{C}$ denote the extended complex plane i.e., $\hat{C} = C \cup \{\infty\}$ is defined as the sequence of points

$$z_0, z_1 = f(z_0), z_2 = f^2(z_0), \cdots, z_n = f^n(z_0) = f(z_{n-1}), \tag{2.1}$$

Here, f^n denote the n[th] iterate of f, that is, f composed with itself n times. The point z_0 is called the seed of the orbit.

For each point $z_0 \in \hat{C}$, we are interested in the behavior of the sequence given in (2.1) and in particular, what happens as n goes to infinity.

Definition 2.2: A point $z_0 \in \hat{C}$ is called *periodic point* of f if $f^n(z_0) = z_0$ for some integer $n \geq 1$. The smallest n with this property is called the period of z_0. Thus, the periodic points of z_0 are the zeros of the function $F(z_0, f) = f^n(z_0) - z_0$.

A periodic point with period one is termed as *fixed point* of f i.e., z_0 is a *fixed point* of f if $f(z_0) = z_0$.

Definition 2.3: Let $v \in \hat{C}$. For any complex valued function $f:D \to D$ where $D \subseteq \hat{C}$, the *attracting basin* or *basin of attraction* of v under the function f is defined as the set $A_f(v)$ of all seed values whose orbit limits to the point v, i.e.

$$A_f(v) = \{z \in D : f^n(z) \to v\}$$

Note that the point v does not necessarily have to lie in D. However, if $v \in D$ and f is continuous then for $a, z_0 \in D$ with $f^n(z_0) \to a$ implies a is a fixed point of f, as

$$f(a) = f\left(\lim_{n\to\infty} f^n(z_0)\right) = \lim_{n\to\infty} f^{n+1}(z_0) = a$$

Fixed points play a major role in the study of dynamical systems. So, it is necessary to give some special attention to the fixed points whenever they arise. Bellow, we discuss some types of fixed points that may arise in dynamical systems.

Definition 2.4 (Attracting Fixed point): Let f be a map from its domain $D \subseteq \hat{C}$ in to itself, then

(i) a finite fixed point $a \in C$ is called an *attracting fixed point* of f if there exist a neighborhood U of a such that the action of f moves any point z in U other than a closer to a, i.e. $|f(z) - a| < |z - a|$

(ii) Suppose, $f(\infty) = \infty$ then ∞ is called an attracting fixed point of f if there exist a neighborhood $U \subset \hat{C}$ of ∞ such that for any point z in $D \cap U$ other than ∞

$|f(z)| > |z|$, i.e. the action of f is to move each point in $D \cap U - \{\infty\}$ closer to ∞.

Remark 2.5: If a is a fixed point of a continuous map f, then there necessarily existing some neighborhood $U \subseteq A_f(a)$. The proof of this does not require that f be differentiable at a, but as we are only interested in specific differentiable functions, here we give a proof only for the case when $|f'(a)|<1$.

Theorem 2.6: Let $f: D \to D$ where $D \subseteq C$ be such that $f(a)=a$, $a \in D$ and $|f'(a)|<1$, then a is an attracting fixed point of f. Furthermore, there exist some $\varepsilon > 0$ such that $S_\varepsilon(a) \cap D \subseteq A_f(a)$.

Proof: Since $|f'(a)| <1$, one can select some $\delta > 0$ such that $|f'(a)| < \delta < 1$. By definition,

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z-a} ,$$

Therefore there exist $\varepsilon > 0$ such that for any $z \in D - \{a\}$ and for which $|z-a| < \varepsilon$, we have

$$\left| \frac{f(z)-a}{z-a} \right| = \left| \frac{f(z)-f(a)}{z-a} \right| < \delta \Rightarrow |f(z) - a| \le \delta\, |z-a| < \varepsilon$$

This shows that for points $z \in S_\varepsilon(a)$, the function f moves z closer to a by a factor of at least δ. Hence, a is an attracting fixed point of f. If we further iterate the map f at z, generating the orbit of z, each application of f takes the corresponding orbit point a step closer to a. Hence, by using induction we may show that

$|f^n(z) - a| \le \delta^n |z-a| \le \delta^n \varepsilon \to 0$ Whenever, $z \in D$ with $|z - a| < \varepsilon$.

Thus $S_\varepsilon(a) \cap D \subseteq A_f(a)$

Remark 2.7: From above it is clear that smaller the value of δ implies faster the convergence of $f^n(z)$ towards a. As $|f'(a)| < \delta$, one can choose the value of δ according as the value of $|f'(a)|$ In particular, if $f(a) = a$ and $f'(a) = 0$, then the value of δ can be taken to be extremely small leading to very fast convergence. Hence, in such a case the fixed point a is called *supper attracting*.

Definition 2.7: (Repelling fixed point) let f be a map from its domain $D \subseteq \hat{C}$ in to itself, then

(i) a finite fixed point $a \in C$ is called an *repelling fixed point* of f if there exist a neighborhood U of a such that the action of f moves any point z in U other than a further from a, i.e. $|f(z)-a| > |z-a|$

(ii) Suppose, $f(\infty)=\infty$ then ∞ is called an *repelling fixed point* of f if there exist a neighborhood $U \subset \hat{C}$ of ∞ such that for any point z in $D \cap U$ other than ∞

$|f(z)| < |z|$, i.e. the action of f is to move each point in $D \cap U - \{\infty\}$ further from ∞.

Theorem 2.8: Let $f : D \to D$ where $D \subseteq C$ be such that $f(a) = a$, $a \in D$ and $f'(a) > 1$, then a is a repelling fixed point of f. Furthermore, there exist some $\varepsilon > 0$ such that for all $z \in D \cap S_\varepsilon(a) - \{a\}$ the orbit $f^n(z)$ eventually leaves, i.e. there exists N such that $f^N(z) \notin S_\varepsilon(a)$.

This theorem can be proving by some quick modification of the proof of the Theorem 2.6.

Definition 2.9: The *multiplier* (or *eigenvalue*, *derivative*) λ of a rational map f iterated n times, at the periodic point z_0 is defined as:

$$\lambda = \begin{cases} f^{n'}(z_0), & if \;\; z_0 \neq \infty \\ \dfrac{1}{f^{n'}(z_0)}, & if \;\; z_0 = \infty \end{cases}$$

Where $f^{n'}(z_0)$ is the first derivative of f^n with respect to z at z_0.

Note that, the multiplier is same at all periodic points of a given orbit. Therefore, it can be regarded as multiplier of the periodic orbit.

The absolute value of the multiplier is called the *stability index* of the periodic point. It is used to check the stability of periodic points.

Definition 2.10: A periodic point z_0 is called *attracting* periodic point if $|\lambda| < 1$, *supper attracting* if $|\lambda| = 0$ and is *repelling* if $|\lambda| > 1$. It is called *indifferent or neutral* when $|\lambda| = 1$

Definition 2.11: A dynamical system f is called *chaotic* if the following three conditions are fulfilled:

- Periodic points of f are *dense,*
- The function f is *transitive*, and
- f Depends *sensitively on initial conditions*.

The periodic points of f are called *dense* if for any periodic point p_1 of f and for any $\varepsilon > 0$, however small may be, the open sphere $S_\varepsilon(p_1)$ contains another periodic point p_2 of f.

The function f is called *transitive* if for any pair of points x and y and for any $r > 0$ there is a third point $z \in S_r(x)$, i.e. the open sphere center at x and radius r, whose orbit comes within the sphere $S_r(y)$. Further, f depends *sensitively on initial conditions* if there is a $R > 0$ such that for any $S_\varepsilon(x)$ there is $y \in S_\varepsilon(x)$ and a positive integer k such that the distance between $f^k(x)$ and $f^k(y)$ is at least R.

To carry out our study for the rest of this paper we consider the maps of the form:

$$f_c(z) = z^2 + c \quad (2.2)$$

For different values of the parameter $c \in \hat{C}$ Below we have given a brief review of the reason for choosing such maps:

Consider, $h(z) = \alpha z + \beta$ where, $\alpha \neq 0$ (2.3)

Now, $h^{-1}(f_c(h(z))) = h^{-1}(\alpha^2 z^2 + 2\alpha\beta z + \beta^2 + c)$

$$= \alpha z^2 + 2\beta z + \frac{\beta^2 - \beta + c}{\alpha}$$

By choosing appropriate values of α, β and c we can make this expression in to any quadratic function f that we please.

Then

$$h^{-1}.f_c.h = f \Rightarrow h^{-1}.f_c{}^k.h = f^k \text{ for all } k$$

This means that the sequence of iterates $\{f^k(z)\}$ of a point z under f is just the image under h^{-1} of the sequence of iterates $\{f_c{}^k(h(z))\}$ of the point $h(z)$ under f_c. The map h transform the dynamical picture of f to that of f_c. In particular, $f^k(z) \to \infty$ if and only if $f_c{}^k(z) \to \infty$. The transformation h is called a conjugacy between f and f_c for some c. Any quadratic function is conjugate to f_c for some c. So, by studying the dynamics of f_c for $c \in C$ we effectively study the dynamics of all quadratic polynomials. Now, we define the *basin of attraction* $B_{f_c}(\infty)$, filled in Julia set $K(f_c)$, Julia set J_c of the map f_c.

Definition 2.12 : The set $K(f_c)$ of all those points of $\hat{C}$ which do not converge to ∞ under iteration of the map f_c is called the *filled in Julia set* of the map f_c i.e.,

$$K(f_c) = C - B_{f_c}(\infty)$$

Clearly $K(f_c)$ is the set of all those points of $\hat{C}$ whose orbits are bounded under iteration of the map f_c.

Definition 2.9 : The Julia set J_c of the map f_c is the *boundary* $\partial K(f_c)$ of the filled in Julia set $K(f_c)$.

Note that, the Julia set J_c separates the two sets filled in Julia set $K(f_c)$ and basin of attraction $B_{f_c}(\infty)$ of infinity. Thus, for each $z_0 \in J_c$, there is an open sphere $S_r(z_0)$ with centre at z_0 and radius $r > 0$, containing a point $u \in S_r(z_0)$ such that iterates of u under f_c converge to infinity as well as another point $v \in S_r(z_0)$ such that iterates of v under f_c do not converge to infinity.

Theorem 2.1 : The filled in Julia set $K(f_c)$ is contained inside the closed disc of radious $\max\{|c|\ ,\ 2\}$. That is $K(f_c) \subseteq \{z : |z| \le \max\{|c|\ ,\ 2\}\}$.

The proof of this theorem is immediate consequence of the following two lemmas.

Lemma 2.2 : If $|c| \le 2$, then the orbit of the points lie outside the circle of radius 2 i.e. the set of points $\{z :\ |z| > 2\}$, escape to infinity.

Note that if $|c| \le 2$, then $K(f_c)$ is a subset of the closed disc with center at 0 and radius 2, i.e. if $|c| \le 2$ then $K(f_c) \subseteq \{z : |z| \le 2\}$.

The proof of this lemma can be found in [11].

Lemma 2.3 : If $|z| \ge |c| > 2$, then $|f_c^n(z)| \to \infty$ as $n \to \infty$ which means that $z \notin K(f_c)$, i.e. $z \in B_{f_c}(\infty)$.

From this lemma it is clear that if $|c| > 2$, then $K(f_c) \subseteq \{z : |z| < |c|\}$.

The proof of this lemma can be found in [5].

3. ROLE OF CRITICAL ORBIT IN DETERMINING THE DYNAMICS OF f_c

The structure of the Julia set is strongly influenced by the behavior of the critical point (see definition 3.1) of f_c. Clearly, f_c has a single critical point at $z = 0$. The orbit of the critical point is called the critical orbit of f_c. The critical point of f_c is the point z for which the pre-images of any given neighborhood of z under f^{-1} are not all distinct and thus, allowing a well defined inverse to be specified.

Let $f(z)$ be an analytic map on C whose power series expansion at $z_0 \in C$ has the form

$$f(z) = f(z_0) + a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \ . \ . \ ., \text{where } a_k \ne 0$$

In this case we called z_0 maps to $f(z_0)$ with degree or multiplicity $v_f(z_0) = k$. Now, z_0 is called a *critical point* of f if $v_f(z_0) > 1$.

From the expansion of $f(z)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = a_k(z - z_0)^{k-1} + a_{k+1}(z - z_0)^k + \ . \ . \ .$$

Since, $k > 1$ taking limit as $z \to z_0$ implies $f'(z_0) = 0$.

Definition 3.1: A point $z_0 \in \hat{C}$ is called *critical point* for the analytic function f if $f'(z_0) = 0$.

3.2 The Schwarzian Derivative: The Schwarzian derivative, named after the German mathematician Hermann

Schwarz, is a strange operator that is invariant under linear fractional transformation. Rather than trying to motivate its origin further, we simply define it and try to explore its properties to find out the role of critical point in the dynamics of polynomial maps like f_c.

Definition 3.3 The *Schwarzian derivative* of a function f is defined as

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\left(\frac{D^2 f(x)}{Df(x)}\right)^2$$

Where, $D^n f(x)$ represent the n^th^ derivative of the function $f(x), \quad n=1,2,3$.

Proposition 3.4: If f is a polynomial of degree at least two such that all roots of its derivative f' are real, then $Sf(x) < 0$. In particular, if all roots of f are real, then $Sf(x) < 0$.

Proof: Suppose, $\alpha_1, \alpha_2, \ldots, \alpha_n$; $n \geq 1$ be the real roots of f', then one can write

$$f'(x) = a(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n).$$

Taking log on both sides we get

$$\log f'(x) = \log a + \sum_{i=1}^{n} \log|x-\alpha_i|$$

Differentiating, $\dfrac{f''(x)}{f'(x)} = \sum_{i=1}^{n} \dfrac{1}{x-\alpha_i}$

Differentiating again, $\dfrac{f'''(x)}{f'(x)} - \left[\dfrac{f''(x)}{f'(x)}\right]^2 = -\sum_{i=1}^{n} \dfrac{1}{(x-\alpha_i)^2}$

Thus,

$$Sf(x) = -\sum_{i=1}^{n} \frac{1}{(x-\alpha_i)^2} - \frac{1}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 < 0$$

For the second part, suppose that the distinct roots of f are $\beta_1 < \beta_2 < \ldots < \beta_k$ where each root β_i is of multiplicity m_i ; $1 \leq i \leq k$. Thus, if d is the degree of f then $\sum_{i=1}^{k} m_i = d$. Applying mean value theorem to f on each of the interval (β_i, β_{i+1}) we can find a root of f' in (β_i, β_{i+1}) for each $i = 1, 2, \ldots, k-1$. Also, f' is divisible by $(x-\beta_i)^{m_i-1}$ for each i . Therefore, f' has at least $(k-1) + \sum_{i=1}^{k}(m_i - 1) = d-1$ real roots. But f' is of degree $d-1$, so, all the roots of f' must be real. Hence as shown above, $Sf(x) < 0$.

Although The Schwarzian derivative does not interact particularly well with most operations on functions e.g., addition, subtraction, division etc., functions with negative Schawarzian derivatives have very interesting dynamical properties that simplify their analysis. The main reason for that is the fact that this property (-ve Schwarzian derivative) is preserved by composition of functions and consequently by iteration of function.

Proposition 3.5: Suppose f and g are two functions and $h = f \circ g$, then

$$Sh(x) = Sf(g(x)).[Dg(x)]^2 + Sg(x).$$

In particular, if $Sf < 0$ and $Sg < 0$ then $Sh < 0$

Proof: From the usual chain rule:

$$Dh(x) = Df(g(x)).Dg(x)$$

$$D^2h(x) = D^2 f(g(x)).[Dg(x)]^2 + Dg(x).D^2g(x)$$

$$D^3h(x) = D^3 f(g(x)).[Dg(x)]^3 + 3D^2 f(g(x)).D^2g(x).Dg(x) + Df(g(x)).D^3g(x).$$

$$\therefore\ Sh(x) = \frac{D^3 f(g(x)).[Dg(x)]^3 + 3D^2 f(g(x)).D^2g(x).Dg(x) + Df(g(x)).D^3g(x)}{Df(g(x)).Dg(x)}$$

$$-\frac{3}{2}\left[\frac{D^2 f(g(x)).[Dg(x)]^2 + Dg(x).D^2g(x)}{Df(g(x)).Dg(x)}\right]^2$$

$$= \left\{\frac{D^3 f(g(x))}{Df(g(x))} - \frac{3}{2}\left[\frac{D^2 f(g(x))}{Df(g(x))}\right]^2\right\}[Dg(x)]^2 + \frac{D^3g(x)}{Dg(x)} - \frac{3}{2}\left[\frac{D^2g(x)}{Dg(x)}\right]^2$$

$$= Sf(g(x)).[Dg(x)]^2 + Sg(x)$$

Now, $Sf < 0 \Rightarrow Sf(g(x)) < 0$ also, $Sg < 0 \Rightarrow Sg(x) < 0$ which implies that $Sh(x) < 0$.

It is difficult to interpret graphically the properties that a function with negative Schwarzian derivative follow. However, by the following proposition we can at least say something about the behavior of its derivative:

Proposition 3.6: If the Schwarzian derivative of the function f is always negative then its derivative cannot have a positive local minimum or a negative local maximum.

Proof: Suppose that x_0 is a local extremum of Df. Which implies that $D^2 f(x_0) = 0$ and hence by the definition of Schwarzian derivative,

$$Sf(x_0) = \frac{D^3 f(x_0)}{Df(x_0)}$$

By our assumption f is of negative Schwarzian derivative, therefore we must have either

$$Df(x_0) > 0\ \ \&\ D^3 f(x_0) < 0 \text{ or } Df(x_0) < 0\ \ \&\ D^3 f(x_0) > 0.$$

Now if Df has a positive local minimum at x_0, then by definition we get $Df(x_0) > 0$ Therefore, $D^3 f(x_0) < 0$. This means that $D^2 f$ changes sign from positive to negative at x_0. This turns out that x_0 is a maximum for Df rather than a minimum. Hence, Df cannot have a positive local minimum.

Similarly, Df cannot have a negative local maximum, for if Df has a negative local maximum at x_0, then $Df(x_0) < 0$ so that $D^3 f(x_0) > 0$. This means that $D^2 f$ changes sign from negative to positive at x_0, meaning that x_0 is a minimum rather than a maximum.

Theorem 3.7: Suppose that Sf is always negative. If x_0 is an attracting periodic points of f, then either the immediate basin of attraction of x_0 extend to $\pm\infty$, or there is a critical orbit of f whose orbit is attracted to the orbit of x_0.

Proof: We will first show that if x_0 is a attracting fixed point with finite immediate attracting basin, then the basin contains a critical point.

The immediate basin of attraction of a fixed point x_0 must be an open interval, for otherwise by continuity, we could extend the basin beyond the end points. Suppose, (a, b) be the immediate basin of attraction of the fixed point x_0 where a and b both finite.

Since $f:(a,b)\to(a,b)$, f must preserve the end points of (a,b) and therefore, $f(a)$ and $f(b)$ are end points of $f[(a,b)]$.

Case 1: When $f(a) = f(b)$, i.e. if $f(a) = a = f(b)$ or $f(a) = b = f(b)$

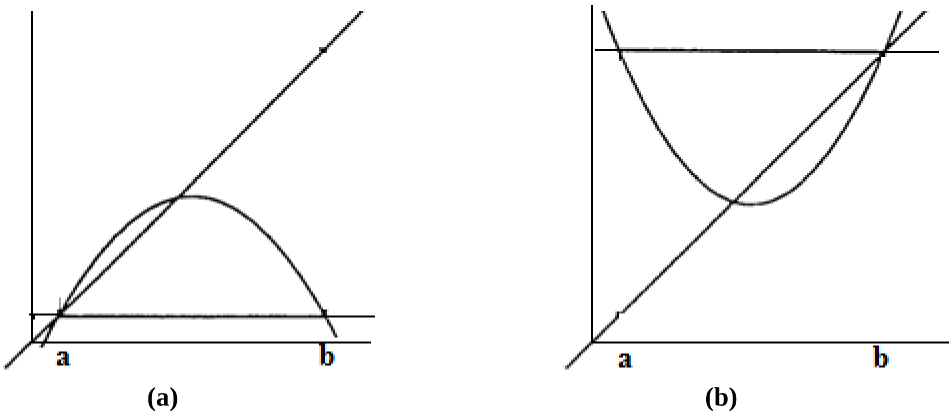


Figure 3.1: (a) $f(a)=a=f(b)$, (b) $f(a)=b=f(b)$

In this case Rolle's theorem implies that there is a point $c \in (a,b)$ such that $Df(c)= 0$,i.e. (a,b) contains a critical point c of f which must be attracted to x_0.

Case 2: Suppose $f(a) = a$ and $f(b) = b$

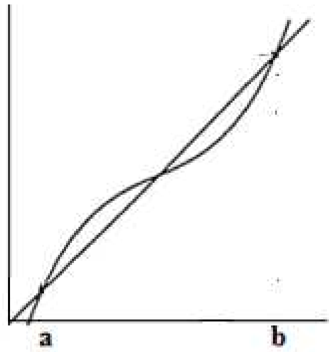


Figure 3.2 ($f(a) = a$ **and** $f(b) = b$ **)**

Since (a,b) is the immediate basin of attraction of x_0 so f can have no other fixed point in (a,b) other than x_0. Clearly, $f(x) > x$ on (a, x_0) and $f(x) < x$ on (x_0, b), since otherwise nearby orbit points would move away from x_0. Now, the mean value theorem implies that there is a $c \in (a, x_0)$ for which

$$Df(c) = \frac{f(a) - f(x_0)}{a - x_0} = \frac{a - x_0}{a - x_0} = 1.$$

Note that $c \neq x_0$ as $Df(x_0) < 1$.

Similarly, there is a point $d \in (x_0, b)$ such that $Df(d)=1$. Therefore, on the interval (c, d) which contains x_0 in its interior, we have $Df(c) = 1$, $Df(x_0) < 1$, and $Df(d) = 1$. So, Df has a local minimum somewhere in (c, d). By Proposition 3.6, Df cannot have a positive local minimum in (c, d) and so it must attain a negative value in (c, d). By intermediate value theorem, Df must take the value 0, and hence f has a critical point in (a, b).

Case 3: When $f(a) = b$ and $f(b) = a$

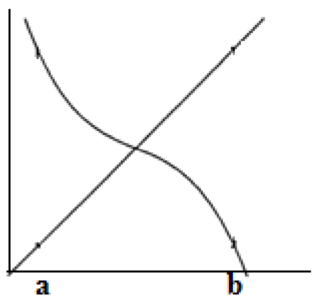


Figure 3.3: ($f(a) = b$ **and** $f(b) = a$ **)**

Suppose $g = f^2$. Clearly (a, b) is the basin of attraction of attracting fixed point x_0 for g. Now, $g(a) = f(f(a)) = f(b) = a$ and $g(b) = f(f(b)) = f(a) = b$. Also by the Proposition 3.5, $Sg < 0$, therefore, by case 2 above, g has a critical point c in (a, b).

Now, $Dg(c) = 0 \Rightarrow Df(f(c)).Df(c) = 0 \Rightarrow Df(f(c)) = 0$ or $Df(c) = 0$

Which follows that one of c or $f(c)$ is a critical point of f. As both of them lie in (a,b), f has a critical point in (a,b).

Note that the periodic points play an important role in the iteration theory and hence it has an important role in non- linear dynamics also. As the critical point will find the attracting cycles of the function used for iteration, an immediate consequence of this theorem is that it helps us to explain why critical points are used to plot orbit diagrams.

An immediate application of this theorem to the map f_c is that as for $|z| > \max\{2,|c|\}$, the orbit of z under f_c will be unbounded. So, there is no infinite basin of attraction. Thus, the second part of the theorem applies since $Sf_c < 0$. Also, the only critical point of f_c is at $z = 0$, therefore, if there is an attracting periodic points for f_c, the orbit of $z = 0$ will find it.

4. THE MANDELBROT SET

Generally, in the study of polynomial function f as a dynamical system, we first choose a seed value z_0 and then try to understand the long term behavior of the sequence: $z_0, \ z_1 = f(z_0), \ z_2 = f(z_1), \ . \ . \ .$ More particularly, we will try to answer such question as:

- For a fixed polynomial f what seed value leads to bounded sequence?
- For a parameterized family of polynomials, how do the set of such seeds depend on the parameter ?

Mainly for most of the families of polynomial systems, there is no obvious picture to make in the parameter space, since there is no obvious question to ask. The study of the dynamics of complex polynomials and rational functions is a success story mainly because of the role played by the critical points, and therefore, in this cases we may study about:

- What happens to the critical points under iterations?

The Mandelbrot set is an answer to this question for the complex one parameter family of quadratic polynomials $\{f_c : c \in C\}$.

The subset of the parameter plane (or *c*-plane) consisting of all parameter value c for which the orbit of the critical point $z = 0$ under the map f_c, i.e.

$$f_c : 0 \to c \to c^2 + c \to \left(c^2 + c\right)^2 + c \to \cdots$$

Is bounded is termed as the Mandelbrot set.

Definition 4.1: The Mandelbrot set M is defined as:

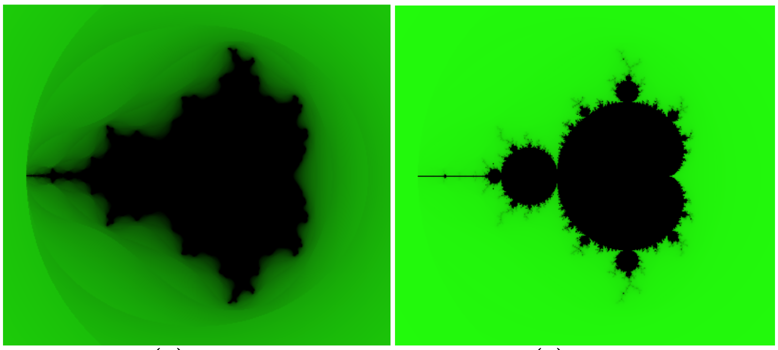
$$M = \{c \in C : \textit{The Orbit of } 0 \textit{ is bounded under iteration by } f_c\}$$

$$= \left\{c \in C : \exists r > 0, \left| f_c^n(0) \right| \leq r, \forall n \in N \right\}$$

One of the particular interests is to represent Mandelbrot set graphically. The simplest algorithm for generating a representation of the Mandelbrot set is known as 'Escape Time' algorithm where we color each point on the parameter space depending on where its attractor lies i.e. whether it is attracted to infinity or bounded within the set.

In view of Lemma 2.3 it is clear that, if $|c| > 2$, then $|f_c(0)| = |c| > 2$ and therefore the orbit of the critical point $z = 0$ necessarily escape to infinity in this case. To distinguish these points we assign a particular color, say color-1, to these points, i.e. the points for which $|c| > 2$. Now, it is clear that Mandelbrot set constituted by some points in the parameter plane within the $[-2, 2] \times [-2, 2]$ grid. To find out these points, we first choose a maximum number of iteration, say N, as the "bailout" limit. Then, for each c value in this grid, we compute the set of points $\{ f_c^n(0) : n = 1, 2, \cdots, N \}$. If for some k $(0 < k < N)$, $f_c^k(0) > 2$, then clearly $|f_c^n(z)| \to \infty$ as $n \to \infty$, and therefore, we stop further iteration and assign color-1 to these points. Again if, $|f_c^k(0)| < 2 \ \forall \ k = 1, 2, \cdots, N$ then we assign another color (contrasting to color-1), say color-2 to this point in the parameter plane. After completing the iteration process for all the c - values the region covered by color-2 is an approximation of the Mandelbrot set M .

It is to be noted that though this algorithm always correctly identifies the points outside the Mandelbrot set, the imposed maximum number of iteration (i.e. N) causes the algorithm to mis-classify points as being inside the set since for starting values very close to but not in the Mandelbrot set may take hundreds or thousands of iterations to escape. This mis-classification of points near the boundary of M set can be notice by comparing two figures given bellow (Figure 4.1) where Figure (a) and (b) are generated by taking $N = 10$ and $N = 150$ respectively and considering 'green' as 'color-1' and 'black' as 'color-2'.



(a) (b)

Figure 4.1: Graphical Representation of Mandelbrot Set *[(A) N=10(B) N=150*

As the Figure $4.1(a)$ is generated by less number of iterations, this figure mis-classified more points as bounded. Thus, higher the value of N, i.e. the number of iterations, will give more detail image of the Mandelbrot set, but in this case the computer will take more time for generating the image.

The Mandelbrot set's true visual beauty rely on the coloring near its boundaries. Developing a strong coloring algorithm helps display the beauty of the set by providing the stunning visual aspect of the set, which also gives the excitement of studying the set? One of the most popular way of doing this is by assigning different colors to the points in the various regions such as inside the set, boundary of the set. Also, for the points just outside the boundary, colors are determined by the number of iterations needed by the point to exceed a certain test value (usually 2). In the Figure 4.2 , we use blue for the points inside the set, green for that in the boundary of the set and orange color for the points just outside the set. Gradually deeper color in orange indicates less number of iterations needed to exceed test value 2 in magnitude.

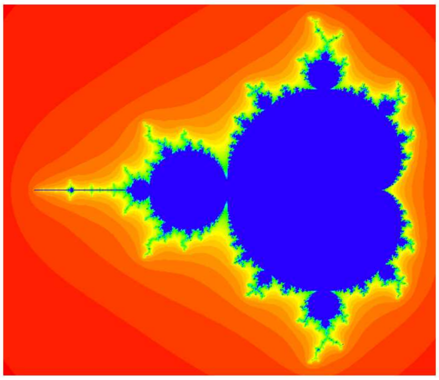


Figure 4.2: (The Mandelbrot Set)

4.2 Some Observations on Mandelbrot Set

Computer images of the Mandelbrot set, as in the $Figure\ 4.2$, shows the estimated geometry of the Mandelbrot set. It contains a big cardioid shaped region, called the *body* of the Mandelbrot set. This region is indicated by B in Figure 4.3 and it intersects the real axis at $c = \frac{1}{4}$ & $c = -\frac{3}{4}$. Towards its left, a circular area H with center at $c = -1$ and radius $\frac{1}{4}$ is attached, called the *head* of the set.

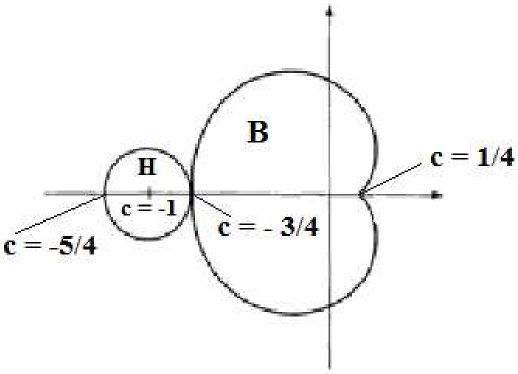


Figure 4 3: (The Body B and Head H of the Mandelbrot Set)

The surface of these two parts are covered by some richly detailed structure of decoration which makes the set a fractal one. Closer inspection of these decorations shows that all of them are different in shape. Any such decoration directly attached to the body is called a *primary bulb* or decoration. In turn, there are many smaller decorations attached to the boundary of each of these decoration as *antennas*. Again, antennas attached to each decoration seems to consist of several *spokes*. The number of such spokes varies from decoration to decoration as clearly visible in the Figure 4.4.

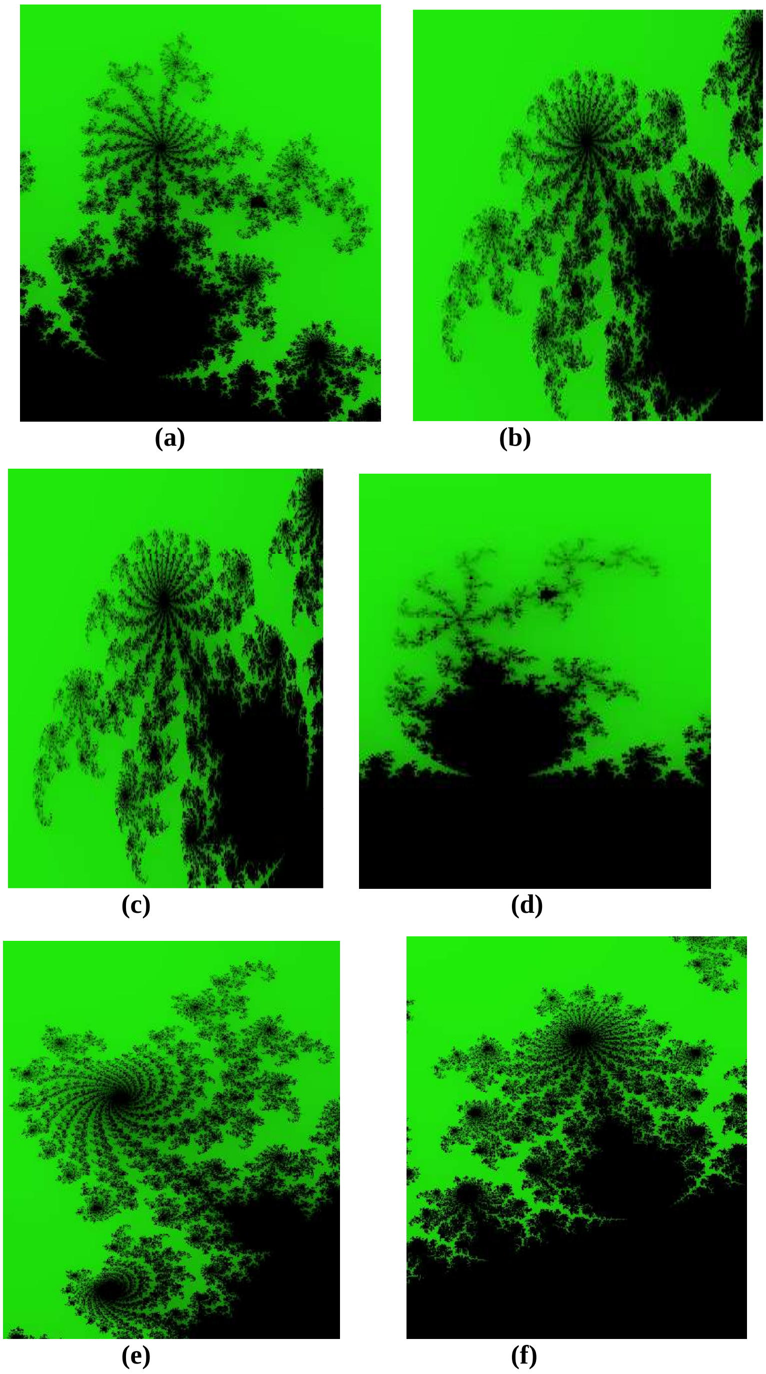
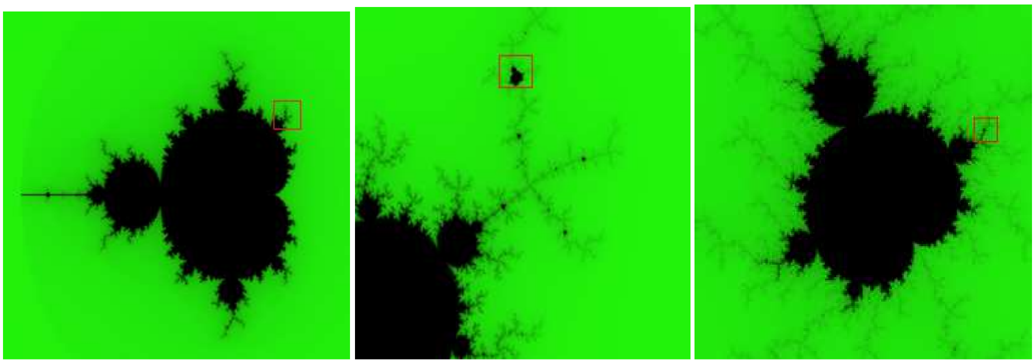
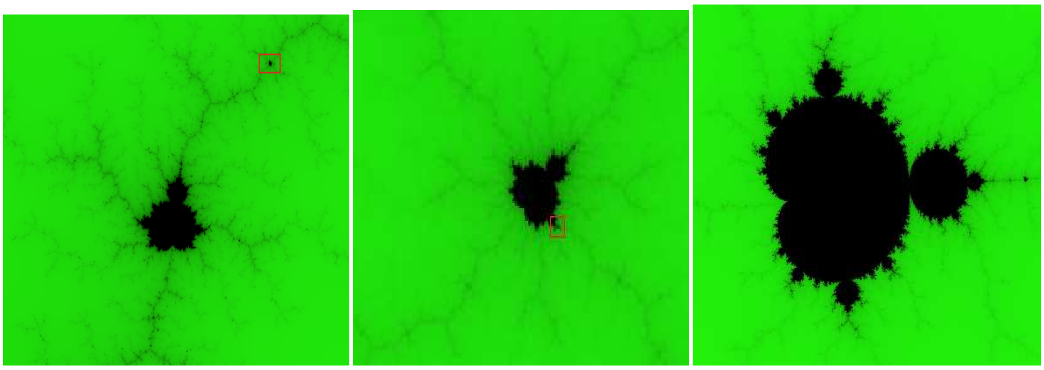


Figure 4.4: (Some Decorations on the Primary Bulb)

The Mandelbrot set is indeed a fractal object, in the sense that it has repeating patterns at different scales. In fact, its boundary is filled with a halo of tiny copies of the entire set, usually referred to as a '*baby Mandelbrot set*'. Each of these baby Mandelbrot set is again surrounded by its own halo of still tinier copies, and so on, smaller and smaller scales without end. For example, Figure 4.4 shows successive magnifications of a portion of the Mandelbrot set. Where the region indicated by a rectangle in the Figure 4.4(a) is magnified and plotted as Figure 4.4(b). Again the portion of the Figure 4.4(b) covered by the rectangle is magnified and plotted as Figure 4.4(c), and so on. Note that, though these Figures certainly suggest that the baby Mandelbrot sets are like island, well separated from one another and from main body of the set, this fact is not true in reality. Douady and Hubbard have shown that the Mandelbrot set is connected [7]. That is, the intricately branched decorations are filled with filaments of baby Mandelbrot set. Though they are not visible at certain level of magnification, they all link back to the main body of the set.



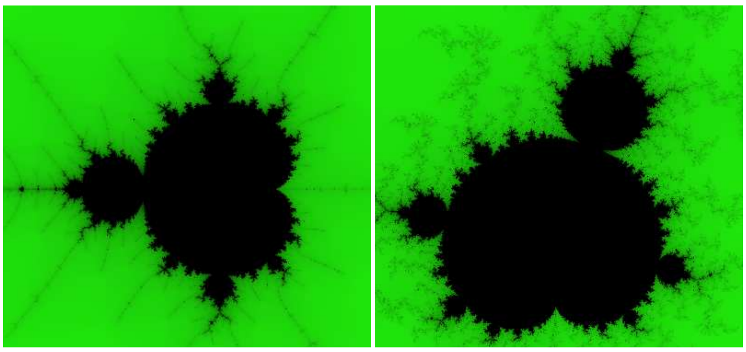
(a) N= 42 **(b) N= 150** **(c) N=210**



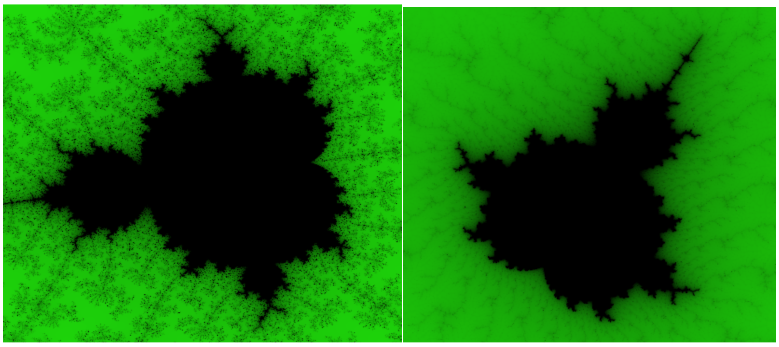
(d) N=425 **(e) N= 900** **(f) N= 9500**

Figure 4.4: (Baby Mandelbrot Set)

Though the Mandelbrot set is a fractal object, its boundary is so complex and intricate that it has a integer dimension [21]. The observation that each filament in the decoration of the Mandelbrot set is filled with baby Mandelbrot set might lead to the wrong conclusion that the Mandelbrot set is self similar. Actually, as Figure 4.5 suggests, every baby Mandelbrot set has its very own pattern of external decorations, everyone different from every other, i.e., the baby Mandelbrot sets are not exact replicas of the full Mandelbrot set. Which lead us to conclude that the Mandelbrot set is not exact self similar although it appears similar to our eyes. Mandelbrot introduced the term 'statistical self similarity' to represent such type of quasi similar objects [13]. Note that, the points constituting these (statistical self-similar) objects belongs to the same statistical distribution.



(a) N = 100 **(b) N = 360**



(c) N= 750 **(d) N= 3400**

Figure 4.5: (Baby Mandelbrot Sets are Not Exact Self Similar)

5. CONCLUSIONS

The Mandelbrot set is one of the most beautiful examples of the fascinating world of fractals. Every little piece of it is loaded with some beautiful almost self similar structures. We tried to explore those beautiful structures. The study can be useful in teaching and learning about the world of fractals.

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